LASU JOURNAL OF MULTIDISCIPLINARY NIGERIAN STUDIES (LASUJMNS)

Vol 2(1) March 2026
ISSN: 3121 9144

REPRESENTATION OF SOLUTIONS OF RADIAL SCHRODINGER EQUATION FOR THE DOUBLE AN-HARMONIC OSCILLATOR VIA TRANSFORMATION TO BICONFLUENT HEUNSEQUATION

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Abstract

In this work, we study the Radia Schrondinger equation corresponding to a double an-harmonic oscillator potential. By means of an appropriate change of variables and a suitable transformation of the wave function, the radial equation is reduced to the canonical form of the biconfluent Heun equation. This transformation enables the representation of exact and quasi-exact solutions in terms of biconfluent Heun functions. We derive the conditions under which polynomial solutions arise, leading to discrete energy spectral and physically acceptable bound states. The obtained results provide a unified analytical framework for treating double an-harmonic oscillator potentials and highlight the relevance of Heun type equations in quantum mechanical systems with non-polynomial potentials. The solution of Radial Schrodinger equation for the double An Harmonic Oscillator via transformation of the Schrodinger equation to Biconfluent Heun's equation. The representation of this solution is explicitly expressed. Our approach offers deeper insight into the spectral properties of the model and facilities further analytical and numerical investigations

Keywords:

Radia Schrondinger equation, Biconfluent Heuns equation, Special functions, Variable transformation, Heuns Functions

Citation:
Awesu B. B., Anjorin A., Owodolu O. J., & Owodolu O. K. (2026). Representation of solutions of radial schrodinger equation for the double an-harmonic oscillator via transformation to biconfluent heunsequation. *LASU Journal of Multidisciplinary Nigerian Studies (LASUJMNS), 2*(1). 17-22

Introduction

The study of exact and quasi-exact solution of the schrondinger equation for an harmonic systems remains a central them in mathematical physics, quantum mechanics, and spectral theory. A harmonic oscilator's arise natural in diverse physical contexts, including molecular vibrations, quantum field theory, semi-conductor physics and effective models of confinement

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ISSN: 3121 9144

in nuclear and particle physics. In particular, the double an harmonic oscillator, characterized by the simultaneous presence of quartic and sextic interaction terms, provides a rich framework for exploring non-perturbative quantum effects beyond the harmonic approximation.

In this work, we investigate the radical Schrondinger equation associated with a double anharmonic oscillator regular singularity and one irregular singularity of rank two at infinity. Its solutions, known as Bicanfluent Heun functions, encompass a wide class of physically relevant wave functions, including those of anharmonic oscilators, coulomb type systems with external fields, and relativistic wave equation in curved space time.

The primary objective of this study is to provide a central representation of the solutions of the radical Schrondinger equation for the double anharmonic oscillator by explicitly transforming it into the Biconfluent Heun equation. This transformation allows the wave functions to be expressed in terms of Biconfluent Heun functions and; under suitable parameter constraints, as finite polynomial solutions corresponding to bound state. Such polynomial solutions arise from termination conditions on the Heun series and yield exact energy eigenvalues.

We begin by reformulating the radial Schrondinger equation through an appropriate change of variables and a gauge-type transformation that removes the dominant asymptotics behavior. This procedure leads to a canomical Biconfluent Heun equation, whose parameters are explicitly related to the physical constraints of the system, including the angular momentum momentum quantum number and the anharmonic coupling strengths. The resulting representation not only clarifies the analytic structure of the solutions but also provides a direct pathway to determining the discrete spectrum.

Beyond its Mathematical elegance, this approach offers several advantages. First, its establishes a unifying framework connecting anharmonic quamtum system with the theory of special functions. Second, it enables the identification of quasi-exactly solvable regimes in which a finite portion of the spectrum can be obtained in a closed form. Finally, it provides a solid foundation for further analytical and numerical investigations, including asymptotic analysis, spectral stability, and perturbative corrections. This work therefore contributes to the growing body of literature on Heun type equations in quantum mechanics, highlighting the central role of the Biconfluent Heun equation in the analytic treatment of higher order anharmonic oscilators and offering new insights into the structure of their radial eigen functions.

Objectives
The primary objective of this study is to develop a clearly analytical framework for representing the solution of the radial Schrondinger equation corresponding to the double anharmonic oscillator potential by transforming the governing differential equation into the Biconfluent Heun equation.

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The objectives of this work are as follows

1. To formulate the radial Schrondinger equation for a quantum particle subject to a double anharmonic oscillator potential of the firm
$V(r) = ar^2 + br^4 - Cr^6 \quad C > 0$
2. To perform a systematic change of variables and parameter rescaling that transforms the reduced radial Schronding equation into the standard form of the Biconfluent Heun Equation and to explicitly identify the correspondence between the physical parameter of the quamtum system and Heun equation parameters.

Methodology

This study adopts an analytical and mathematical-physics base methodology to represent the solutions of the radial Schrondinger equation for a double anharmonic oscillator through transformation to the Biconfluent Heun equation. The procedure is systematic and proceeds through the following stages;

i. Formulation of the Radial Schrondinger equation

We begin with the time-independent Schrondinger equation in three-dimensional space for a particle of mass (m) subject to a central potential. By separation of variables in spherical coordinates, the radial part of the wake function satisfies

$$\frac{d^2 R(r)}{dr^2} + \frac{2dR(r)}{r\ dr} + \left(\frac{2m}{h^2} (E - V(r) - \frac{\ell\ (\ell+1)}{r^2} \right) R(r) = 0$$

The double anharmonic oscillator potential is introduced as $V(r) = ar^2 + br^4 + Cr^6 \quad C > 0$

ii. Reduction to a Schrondinger-type radial equation

To eliminate the first-order derivatives term, the transformation

$$R(r) = \frac{U\ (r)}{r}$$

$$\frac{d^2\ U(r)}{dr^2} + \left(\frac{2m}{h^2} (E - ar^2 - br^4 - cr^6) - \frac{(-1)}{r^2} \right) U(r) = 0$$

iii. Transformation to the Biconfluent Heun equation

After simplification, the resulting equation is brought into the Canonical form of the Biconfluent Heun equation

$$\frac{d^2y}{dx^2} + (\frac{1 + \alpha_1}{z} \ldots \beta_1 - b_1 - 2z\ )\ \frac{dy}{dx} + (r_1 - \ \alpha_1 - 2\text{-}1\ \frac{(\delta_1 + \alpha_1)}{2z}\ \beta_1$$

This establishes a direct correspondence between the radial Schrondinger equation for the double anharmonic oscillator and the Biconfluent Heun equation

Analytical Validation and Consistency

The obtained solution are verified by;

i. Substitution into the original radial Schrondinger equation
ii. Verification of boundary conditions and square – integrability

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iii. Examination of limiting cases (e.g. reduction to the harmonic oscillator when anharmonic terms vanish).

The Bio-confluent Heun's equation (BHE) given by

$$xy'' + (1+\alpha\beta\ \ x - 2x^2)y' + \left((\gamma - \alpha - 2)x - \frac{1}{2} + [\delta + (1+\alpha\beta\ \ ]\right) y = 0. \quad (1)$$

has two singular points at $x = 0$, and $x = \infty$. That of $x = 0$ is regular while that $x = \infty$ is irregular. This equation is one of the confluent of the general Heun's equation.

$$y'' + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-d}\right) y' + \frac{\beta\ \ x - q}{x(x-1)(x-d)} y = 0 \quad (2)$$

Equation (2) has four singular points, three of which are regular at $x = 0, 1, d, \infty$ with exponent $\{0,1-\}, \{0,1-\}, \{0,1-\}$ and $\{\alpha\beta\}$ respectively, ($q \neq 0$). The parameters of (2) are arbitrarily constants connected by the Fushian relation

$1 + \alpha + \beta + 1 = \gamma + \delta + \varepsilon.$

The Radial Schrodinger Equation for the double an-harmonic oscillator given by [**4**]:

$$y''(x_1) + \{E - \mu x_1^2 - \lambda\mu x_1^4 - \lambda x_1^6\} y(x_1) = 0, \quad x_1 \in \mathbb{R}. \quad (3)$$

with corresponding potential

$$v(x_1) = \mu x_1^2 + \mu x_1^4 + \eta x_1^6, \quad \eta > 0. \quad (4)$$

Equations (1), (2), (3) has many applications in Mathematical Physics, Black hole theory, Sothern-Moi Models—just to say but few. They have been the subject of investigation for several years by some authors such asA. Anjorin, K. Koiken, Ronveaux. Just to mention but few.

Using the transformation

$$y(x_1) = x^{1/2} e^{\frac{1}{4}acx_1^4 + \frac{b}{2}\ \ cx_1^2} \psi(x), \quad x_1^2 = \left(\frac{2}{ac}\right)^{1/2} x, \quad (5)$$

Equation (3) reduces to a BCE form [8].

$$x\psi'(x) + \left(\frac{3}{2} + \beta\ \ _c \left(\frac{2}{ac}\right)^{1/2} x - 2x^2\right) \psi(x) \quad (6)$$

.

$$+ \left\{ \frac{1}{2ac}(\beta\ \ _c^2 - 3ac - \mu)x + \frac{1}{4}\left(\frac{2}{ac}\right)^{1/2} (E + \beta\ \ _c) \right\} \psi(x) = 0$$

(6) (7)

Comparing (6) with (1)

We observe the followings.

$$\alpha = \frac{1}{2}, \ \beta\ \ = \beta\ \ _c \left(\frac{2}{ac}\right)^{1/2}, \quad \gamma = \frac{1}{2ac}(\beta\ \ _c^2 - 3ac - \mu) + \frac{3}{2}, \quad \delta = \frac{E}{(2ac)^{1/2}}$$

(7) (8)

Then the solution of Schr¨odinger equation **(3)** can be expressed in terms of the solution of biconfluentHeun's equation when the parameters are specified. Thus,

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$$y(x_1) = B\left(\frac{1}{2}, \beta_c\left(\frac{2}{ac}\right)^{1/2}, \frac{1}{2ac}(\beta_c^2 - 3ac - \mu) + \frac{3}{2}, \frac{E}{(2ac)^{1/2}}, x\right)$$

(8) (9)

The equation of Radial Schro¨dinger equation concerned is

$$\eta''(r) + \left(\frac{\lambda_m + 1/2}{a_m} - \frac{1}{4a_m^2(r-1)} - \frac{l_m(l_m+1)}{r^2}\right)\eta(r) = 0$$

(3) (10)

where $0 \leq t < \infty$, is the eigenvalue, l_m is the rotational quantum number and$\alpha_m > 0$.

Derivative [8, 3, 5, 4] is a coupling parameter using the transformation [8, 12]

$$\eta(r) = \tau_{(l_m+1)} e^{\frac{(r-1)}{2a_m}} \chi(x, r = (\Delta l_m + 2)^{1/2} x)$$

(4) (11)

then equation (3) reduces to

$$xx'(x) + \left\{2l_m + 2 + \left(\frac{2}{a_m}\right)^{1/2} x - 2x^2\right\} x'(x)$$

$$+ \left\{2(2l_m - l_m - 1)^{1/2}(l_m + 1)\right\} x(2) = 0$$

(5) (12)

The reduction of the radial Schrondinger equation to the biconfluent Heun form allows the solutions to be expressed in terms of biconfluent Heun functions. Furthermore, this transformation highlights the fundamental role of Heun-type equations in the analytical treatment of quantum systems with an-harmonic interactions.

Results

Comparing (4) and (2) together, we observe that

$$\alpha = 2l_m + 1; \quad \beta = 1\left(\frac{2}{a_m}\right); \quad \delta = 1 + 2\lambda_m; \quad \gamma = 0$$

If $\eta(r) = B(\alpha, \beta, \gamma, \delta: \tau)$ where $\alpha, \beta, \gamma,$ and δ are given in (6).

Recommendations

Based on the analytical framework developed for representing the solutions of the Radial Scheondinger equation for the double anharmonic oscillator via transformation to the Biconfluent Heun equation, the following recommendations are proposed for future research and further development.

1. Extension to Higher-Order Anharmonic Potentials

It is more recommended that the present methodology be extended to radial Schrondinger equations involving higher order polynomial potentials such as octive or decatic anharmonic

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oscilators. Such extension may lead to generalized Heun-type equations and provide deeper insight into the structure of quasi-exactly solvable quantum systems.

2. Systematic Study of Quasi-Exact Solution

Further investigation is recommended into the quasi-exactly solvable regimes associated with the Biconfluent Heun equation. A systematic classification of parameter values yielding polynomial solutions would enhance understanding of partial spectral analysis and its physical implications.

3. Numerical-Analytical Hybrid Approach

Analytical solutions obtained in terms of Biconfluent Heun functions may be complemented by high precision numerical methods. Such hybrid approaches can be used to validate analytical spectra, study non-polynomial states, and explore parameter conditions are not satisfied.

4. Application to Physical and Inter Disciplinary Models

It is recommended that the derived solution be applied to concrete physical systems, including molecular vibrational spectra, quantum and effective confinement models in nuclear and particle physics. Such applications would enhance the physical relevance of the Biconfluent Heun representation.

Concluding Remarks

The representation of the solution of the Radial Schr¨odinger equation for the Harmonic Oscillator has been shown and explicitly expressed. Other forms of the Radial Schro¨dinger equations are being investigated.

References

P. C. Hansen. Regularization tools for ill-posed problems. *Numerical Algorithms*, 6(1):1–35, 1994.

L.D. Landau and E.M. Litfshitz (1977): Quantum Mechanics: Non-Relativistic Theory; 3rd Ed. Pergomon Press, Oxford.

S.Flugge, (1999): Practical Quantum Mechanics. Springer-Verlag, Berlin.

A. Ushvervidze (1994): Quasi-Exactly Solvable Models in Quantim Mechanics. IOP Publishing, Bristol.

M. A. Shifman (1989): “New Findings in Quantum Mechanics (Partial Algebraization of the Spectral Problem);” Int. J. Mod. Phys. A, 4, 2897 – 2952.

A. Ronveaux (Ed) (1995). Henu’s Differential Equations, Oxford University Press, Oxford.

S.Y. Slavyanov and W. Lay (2000): Special Functions: Unified Theory Based on Singularities, Oxford University Press, Oxford.

C.M. Bender and S. Boethcher (1998)” “Quasi-Exactly Solvable Quartic Potentials”; J. Phys. A; Math. Gen. 31, L273 – L277.

H. Exton (1978): Handbook of Hypergeometric Integrals, Ellis Horwood, Chichester.